MHV gravity amplitudes & their COMBINATORICS

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outline

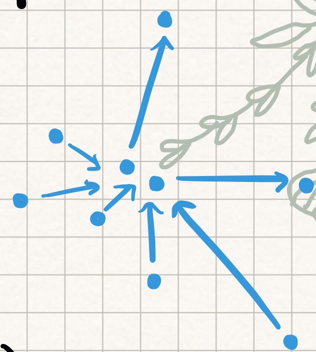
I. What are scattering amplitudes?

II. Spinor-helicity varieties: a mathematical framework + a conjecture

III. Computational evidence

IV. Toward a "Gravituhedron" geometry

I. A Scattering Story



$1, \dots, n$ (massless) particles

kinematic data $\left[\begin{array}{l} \lambda = (\lambda_1, \dots, \lambda_n) \in Gr_2(\mathbb{C}^n) \\ \tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_n) \in Gr_2(\mathbb{C}^n) \end{array}\right.$

$\Longrightarrow P_i := \lambda_i \tilde{\lambda}_i^T \in Mat_2(\mathbb{R})$ external momentum of particle i

spinor variables $\left[\begin{array}{l} \langle ij \rangle := \det(\lambda_i \mid \lambda_j) \\ [ij] := \det(\tilde{\lambda}_i \mid \tilde{\lambda}_j) \end{array}\right.$

PROPERTIES

(i) $\langle ij \rangle = -\langle ji \rangle$ and $[ij] = -[ji]$ $(\langle ii \rangle = [ii] = 0)$

(ii) Schouton identies (Plücker relations)

$$\langle ij \rangle \langle km \rangle - \langle ik \rangle \langle jm \rangle + \langle im \rangle \langle jk \rangle = 0$$
$$[ij][km] - [ik][jm] + [im][jk] = 0$$

(iii) Momentum conservation

$$\sum_{i=1}^{n} P_i = 0 \Longleftrightarrow \sum_{i=1}^{n} \langle ji \rangle [ik] = 0 \quad \forall j,k$$

We denote $s_{ij} := \langle ij \rangle [ij]$ the "Mandelstam invariants"

<u>Scattering amplitudes</u> are analytic functions which compute the probabilities of certain scattering patterns.

① Gluon scattering amplitude (Yang-Mills)

<u>Theorem</u> [Parke-Taylor '86, Berends-Giele ~'86]
The (tree-level) MHV amplitude for n gluon scattering is given by:

$$\mathcal{A}_n^{(0)} = \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \quad (*)$$

Remarks:

(i) Tree-level MHV gluon amplitude is fixed by its (simple, logarithmic) poles; its numerator is trivial

(ii) We can associate $(*)$ to the canonical form on $Gr_{\geq}(2,n)$

$$\Omega(Gr_{\geq 0}(2,n)) := \underbrace{\Omega(\mathcal{M}_{0,n})}_{\frac{dz_2 \cdots dz}{(z_2-z_1)(z_3-z_2)\cdots(z_1-z_n)}} \wedge \underbrace{\Omega(\text{torus})}_{\frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}}$$

This mediates a fruitful dialogue

Grassmanian combinatorics + geometry $\xleftrightarrow{\text{Amplituhedron}}$ gluon scattering amplitudes + on-shell diagrams

MATHEMATICS | PARTICLE PHYSICS

② Graviton scattering amplitude
($\mathcal{N}=8$ SUGRA)

Question: Is there a similar underlying mathematical structure ?

Theorem [Hodges, 2012]
The (tree-level) MHV amplitude for n graviton scattering is:

$$\mathcal{M}_n^{(0)} = \frac{\det \Phi_C^R}{(R)(C)}$$

Φ_C^R: R ← rows R deleted, C ← columns C deleted

$(R) = \langle ab\rangle\langle bc\rangle\langle ac\rangle$

where $\Phi_{ij} = \begin{cases} [ij] & i\neq j \\ -\sum\limits_{\substack{1\leq k\leq n \\ k\neq i}} \frac{[ik]\langle xk\rangle\langle yk\rangle}{\langle ik\rangle\langle xi\rangle\langle yi\rangle} & i=j \end{cases}$ for fixed x, y "reference spinors"

and $R=\{a<b<c\}$, $C=\{d<e<f\}$.

Denote N_n the numerator of $\mathcal{M}_n^{(0)}$.

<u>Example</u>: Let $n=5$, $R=\{1,2,3\}$, $C=\{3,4,5\}$.

$$\det \Phi^R_C = \begin{vmatrix} \frac{[14]}{\langle 14\rangle} & \frac{[24]}{\langle 24\rangle} \\ \frac{[15]}{\langle 15\rangle} & \frac{[25]}{\langle 25\rangle} \end{vmatrix}$$

$$= \frac{[14][25]\langle 24\rangle\langle 15\rangle - [24][15]\langle 14\rangle\langle 25\rangle}{\langle 14\rangle\langle 15\rangle\langle 24\rangle\langle 25\rangle}$$

$$\Longrightarrow N_5 = [14][25]\langle 24\rangle\langle 15\rangle - \langle 14\rangle\langle 25\rangle[24][15].$$

Fact: N_n vanishes at $\langle ij\rangle = [ij] = 0$ $\forall ij$.

Remarks:

(i) Tree-level MHV gravity amplitude is NOT fixed by its poles (double poles in "soft limits", non-logarithmic poles) and the numerator is NOT trivial.

(ii) No connection to logarithmic differential forms of the Amplituhedron geometry...

II. Spinor-helicity varieties and ideals.

Fix two copies of the complex Grassmanian $Gr_2(\mathbb{C}^n)$, subvariety in $\mathbb{P}^{\binom{n}{2}-1}$, with Plücker variables $\langle ij\rangle, [ij]$.

$$R_n = \mathbb{C}[\{\langle ij\rangle, [ij]\}]$$

$$I_n = \left(\left\{ \underbrace{\sum_{i=1}^{n} \langle ji\rangle[ik]=0}_{n^2} \;\middle|\; j,k \right\} \cup \left\{ \begin{matrix} \overbrace{\langle ij\rangle\langle km\rangle - \langle ik\rangle\langle jm\rangle + \langle im\rangle\langle jk\rangle}^{\binom{n}{4}} \\ \underbrace{[ij][km] - [ik][jm] + [im][jk]}_{\binom{n}{4}} \end{matrix} \right\} \right)$$

$$J_{ij} = (\langle ij\rangle, [ij])$$

$$\pi_n : R_n \to R_n / I_n =: Q_n$$

$$J := \bigcap_{1\le i<j\le n} \pi_n(J_{ij}) \subseteq Q_n$$

"Spinor helicity variety"

$$SH(2,n,0) = \{(\lambda, \tilde{\lambda}) : \operatorname{rank}(\lambda) = \operatorname{rank}(\tilde{\lambda}) = 2,\ \operatorname{rank}(\lambda\tilde{\lambda}^T) = 0\}$$
$$\subseteq Gr_2(\mathbb{C}^n) \times Gr_2(\mathbb{C}^n) \subseteq \mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{2}-1}$$

$$SH(2,n,0) \cong Fl(2, n-2; \mathbb{C}^n)$$ [Maazouz, Pfister, Sturmfels, 2024]

2-step flag variety

We equip R_n with a $\mathbb{Z}^{n+2}$-grading

$$(e_{\langle\cdot\cdot\rangle}, e_{[\cdot\cdot]}, e_1, \dots, e_n)$$

$\#\langle\cdot\cdot\rangle$ $\quad$ $\#[\cdot\cdot]$ $\quad$ # 1's $\begin{cases} +1 \text{ in } \langle\cdot\cdot\rangle \\ -1 \text{ in } [\cdot\cdot] \end{cases}$ $\quad$ # n's $\begin{cases} +1 \text{ in } \langle\cdot\cdot\rangle \\ -1 \text{ in } [\cdot\cdot] \end{cases}$

which descends to Q_n.

Fact: $N_n \in Q_n$ lives in multidegree

$$d(n) := \left(\frac{n^2-3n-6}{2}, n-3, n-5, \dots, n-5\right) \in \mathbb{Z}^{n+2}$$

$\hookrightarrow d(5) = (2,2,0,0,0,0,0)$

<u>Conjecture</u>: For $n \geq 5$, $\dim_{\mathbb{C}}(J_{d(n)}) = 1$.

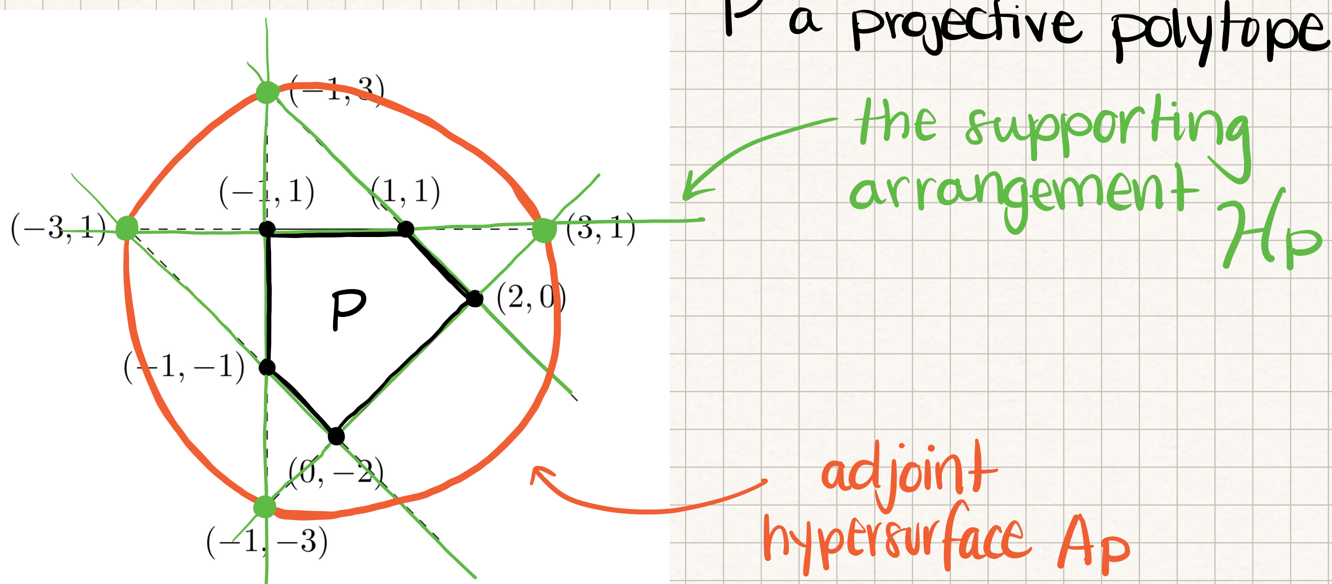
multidegree-$d(n)$ part

We propose a proof by induction.

<u>Upshot</u>: Control over the expression and behavior of N_n.

A note on

ADJOINT HYPERSURFACES



[Gaetz, Positive Geometries Learning Seminar, Lecture 3]

Theorem [Kohn and Ranestad 2019]

$\mathcal{H}_P$ simple $\longrightarrow$ A_P unique

Theorem [Lam, 2022] : $\Omega(P) = \alpha \cdot \dfrac{A_P}{\prod_{\text{facets } H} H} dx$

Question: Can the geometry of A_P help elucidate a geometry underlying MHV gravity amplitudes? Could our conjecture and methods furnish a novel approach to investigating adjoints?

III. Computational Evidence

Confirmed: $n=5$, $n=6$ ✓

- We implement a procedure in Macaulay2.

1. Compute standard monomial basis of $(Q_n)_{d(n)}$ w.r.t. graded reverse lexicographic order.

$$|B_n| = \begin{cases} 16 & n=5 \\ 780 & n=6 \end{cases}$$

$$(\sim 10^7 \text{ for } n=7)$$

2. Solve a linear system of equations.

- Enumerate basis of $(R_n)_{d(n)}$ combinatorially via degrees of Mandelstam invariants.

$\boxed{n=5}$ $\langle\cdots\rangle\langle\cdots\rangle[\cdots][\cdots]$

degree 2
$$\begin{cases} s_{ij}^2 = \langle ij\rangle^2[ij]^2 \Longrightarrow \binom{n}{2} \\ s_{ij}s_{ik} = \langle ij\rangle[ij]\langle ik\rangle[ik] \Longrightarrow \binom{n}{2}\cdot\binom{n-2}{1} \\ s_{ij}s_{k\ell} = \langle ij\rangle[ij]\langle k\ell\rangle[k\ell] \Longrightarrow \binom{n}{2}\cdot\binom{n-2}{2} \end{cases}$$

degree 0
$$\left\{ \langle ij\rangle\langle k\ell\rangle[ik][j\ell] \Longrightarrow 2\binom{n}{2}\binom{n-2}{2} \right.$$

$\boxed{n=6}$ $\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle[\cdots][\cdots][\cdots]$

$\boxed{n=7}$ $\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle\langle\cdots\rangle[\cdots][\cdots][\cdots][\cdots]$

Impediments for $n \geq 7$:

(i) superexponential complexity

(ii) combinatorial description in terms of s_{ij} infeasible

IV. Toward a "Gravituhedron" geometry

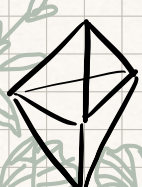
Motivating Question: Can we realize an analogous "Amplituhedron-like" geometry in gravity?

State of the Art: [Trnka, 2020]
arXiv: 2012.15780

R-invariants in Yang-Mills $\longleftrightarrow$ G-invariants in Supergravity

$\updownarrow$?

$\Omega \sim \frac{dx}{x}$



(tree-level) Amplituhedron

?

(tree-level) Gravituhedron

$\Omega \sim ?$